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# Operator Exponentials on Hilbert Spaces

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## Abstract

Let  $\mathcal{H}$  be a complex Hilbert space and let  $\mathcal{L}(\mathcal{H})$  be the Banach algebra of all bounded linear operators on  $\mathcal{H}$ . In this paper we consider the following class of operators:

$$\hat{\Sigma}(\mathcal{H}) = \{S \in \mathcal{L}(\mathcal{H}): S \text{ is a scalar type operator and } \sigma(S) \cap \sigma(S + 2k\pi i) \subseteq \{k\pi i\} \text{ for } k = 1, 2, \dots\}.$$

The main results of this paper read as follows:

1. If  $T, S \in \hat{\Sigma}(\mathcal{H})$  and  $e^T e^S = e^S e^T$  then  $T^2 S^2 = S^2 T^2$ .
2. If  $S \in \hat{\Sigma}(\mathcal{H})$ ,  $T \in \mathcal{L}(\mathcal{H})$  and  $e^T = e^S$  then  $TS^2 = S^2 T$ .

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## 1 Terminology and results

Throughout this paper let  $\mathcal{H}$  denote a complex Hilbert space and  $\mathcal{L}(\mathcal{H})$  the Banach algebra of all bounded linear operators on  $\mathcal{H}$ . For  $A \in \mathcal{L}(\mathcal{H})$  the spectrum and the spectral radius of  $A$  are denoted by  $\sigma(A)$  and  $r(A)$ , respectively. The set of eigenvalues of  $A$  is denoted by  $\sigma_p(A)$ . For the resolvent set of  $A$  we write  $\rho(A)$ . We use  $N(A)$  and  $A(\mathcal{H})$  to denote the kernel and the range of  $A$ , respectively.

An operator  $S \in \mathcal{L}(\mathcal{H})$  is called a *scalar type operator* if  $S$  admits a representation

$$S = \int_{\sigma(S)} \lambda E(d\lambda),$$

where  $E(d\lambda)$  denotes integration with respect to a spectral measure  $E(\cdot)$  on  $\mathcal{H}$ . See [1], [2] and [14] for properties of spectral measures and scalar type operators.

If  $A \in \mathcal{L}(\mathcal{H})$  is *normal* ( $AA^* = A^*A$ ) then  $A$  is a scalar type operator and the values of the spectral measure of  $A$  are selfadjoint projections (see [1], Theorem 7.18).

J. Wermer [14] has shown that the scalar type operators on  $\mathcal{H}$  are those operators which are similar to normal operators. More precisely, Wermer has shown that for every finite set  $S_1, \dots, S_n$  of commuting scalar type operators on  $\mathcal{H}$  there is a selfadjoint operator

$B \in \mathcal{L}(\mathcal{H})$  with a bounded everywhere defined inverse such that the operators  $BS_iB^{-1}$ ,  $i = 1, \dots, n$ , are all normal.

We write  $\Sigma(\mathcal{H})$  for the class of all scalar type operators on  $\mathcal{H}$ . In the present paper we consider the following class of operators:

$$\hat{\Sigma}(\mathcal{H}) = \{S \in \Sigma(\mathcal{H}) : \sigma(S) \cap \sigma(S + 2k\pi i) \subseteq \{k\pi i\} \text{ for } k = 1, 2, \dots\}.$$

Now we state the main results. Proofs will be given in Section 3, in Section 4 we present some corollaries.

**Theorem 1.1** *If  $T \in \hat{\Sigma}(\mathcal{H})$ ,  $S \in \mathcal{L}(\mathcal{H})$  and  $e^Te^S = e^Se^T$  then  $e^ST^2 = T^2e^S$ . If in addition  $\sigma_p(T) \cap \{k\pi i : k = 1, 2, \dots\} = \emptyset$  then  $e^ST = Te^S$ .*

**Theorem 1.2** *If  $T, S \in \hat{\Sigma}(\mathcal{H})$  and  $e^Te^S = e^Se^T$  then  $T^2S^2 = S^2T^2$ .*

**Theorem 1.3** *Suppose that  $T, S \in \hat{\Sigma}(\mathcal{H})$  and that  $e^Te^S = e^Se^T$ .*

- (a) *If  $\sigma_p(T) \cap \{k\pi i : k = 1, 2, \dots\} = \emptyset$  then  $TS^2 = S^2T$ .*
- (b) *If  $\sigma_p(T) \cap \{k\pi i : k = 1, 2, \dots\} = \sigma_p(S) \cap \{k\pi i : k = 1, 2, \dots\} = \emptyset$  then  $TS = ST$ .*

For related results concerning the equation  $e^Ae^B = e^Be^A$  see [10], [11], [12] and [15].

**Theorem 1.4** *Suppose that  $T, S \in \mathcal{L}(\mathcal{H})$ ,  $T + S \in \hat{\Sigma}(\mathcal{H})$  and that*

$$e^{T+S} = e^Te^S = e^Se^T.$$

*If  $\sigma_p(T + S) \cap \{k\pi i : k = 1, 2, \dots\} = \emptyset$  then  $TS = ST$ .*

**Theorem 1.5** *If  $S \in \hat{\Sigma}(\mathcal{H})$ ,  $T \in \mathcal{L}(\mathcal{H})$  and  $e^T = e^S$  then  $TS^2 = S^2T$ . If in addition  $\sigma_p(S) \cap \{k\pi i : k = 1, 2, \dots\} = \emptyset$  then  $TS = ST$ .*

For related results concerning the equation  $e^A = e^B$  see [3], [9] and [11].

## 2 Preparations

In this section we collect some results which we need for the proofs of the theorems in Section 1.

**Proposition 2.1** *Suppose that  $A \in \mathcal{L}(\mathcal{H})$  is normal.*

- (a) *If  $\mu \in \mathbb{C}$  then  $(A - \mu)(\mathcal{H}) = (A^* - \bar{\mu})(\mathcal{H})$ .*

(b) If  $B \in \mathcal{L}(\mathcal{H})$  then

$$E(\sigma(A) \cap \sigma(B))(\mathcal{H}) = \bigcap_{\lambda \in \rho(B)} (A - \lambda)(\mathcal{H}),$$

where  $E(\cdot)$  denotes the spectral measure of  $A$ .

*Proof.* (a) Since  $A$  is normal,  $A - \mu$  is normal. Exercise 12.36 in [8] gives the result. (b) is shown in [7, Theorem 1], see also [6]. ■

Let  $A \in \mathcal{L}(\mathcal{H})$ . The map  $\delta_A : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ , defined by

$$\delta_A(C) = CA - AC \quad (C \in \mathcal{L}(\mathcal{H}))$$

is called the *inner derivation* determined by  $A$ . It is clear that  $\delta_A$  is a bounded linear operator on  $\mathcal{L}(\mathcal{H})$  with  $\|\delta_A\| \leq 2\|A\|$ .

Throughout this paper let  $f$  denote the entire function  $f : \mathbb{C} \rightarrow \mathbb{C}$  given by

$$f(z) = \begin{cases} z^{-1}(e^z - 1), & \text{if } z \neq 0, \\ 1, & \text{if } z = 0. \end{cases}$$

Let  $M_A = \{\lambda \in \sigma(\delta_A) : f(\lambda) = 0\}$ .

**Proposition 2.2** *Let  $A \in \mathcal{L}(\mathcal{H})$ .*

- (a) *If  $M_A = \emptyset$ , then  $f(\delta_A)$  is an invertible operator on  $\mathcal{L}(\mathcal{H})$ .*
- (b) *If  $\lambda \in M_A$  then  $\lambda$  is a simple zero of  $f$  and there is  $j \in \mathbb{Z} \setminus \{0\}$  with  $\lambda = 2j\pi i$ .*
- (c)  *$M_A$  is a finite set,  $M_A \subseteq \{\pm 2\pi i, \pm 4\pi i, \dots\}$ .*
- (d) *If  $M_A \neq \emptyset$  and  $M_A = \{\lambda_1, \dots, \lambda_p\}$  with  $\lambda_j \neq \lambda_k$  for  $j \neq k$  then*

$$N(f(\delta_A)) = N(\delta_A - \lambda_1) \oplus \dots \oplus N(\delta_A - \lambda_p).$$

- (e)  $\sigma(\delta_A) = \{\lambda - \mu : \lambda, \mu \in \sigma(A)\}$ .
- (f)  $e^{\delta_A}(C) = e^{-A}Ce^A$  for all  $C \in \mathcal{L}(\mathcal{H})$ .
- (g)  $f(\delta_A)(\delta_A(C)) = e^{-A}Ce^A - C$  for all  $C \in \mathcal{L}(\mathcal{H})$ .

*Proof.* (a) If  $M_A = \emptyset$ , then  $f(\lambda) \neq 0$  for all  $\lambda \in \sigma(\delta_A)$ , thus  $f(\delta_A)$  is invertible.

(b), (c) and (d) are shown in [11].

(e) follows from [4], and Proposition 6.4.8 in [5] shows that (f) holds.

(g) follows from (f) and  $zf(z) = f(z)z = e^z - 1$ . ■

**Proposition 2.3** *Let  $A$  be a normal operator in  $\mathcal{L}(\mathcal{H})$  and let  $E(\cdot)$  be its spectral measure. If  $\lambda_0 \in \mathbb{C}$ ,  $C \in N(\delta_A - \lambda_0)$ ,  $D \in N(\delta_A + \lambda_0)$  then*

$$C(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A - \lambda_0))(\mathcal{H})$$

and

$$D^*(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A - \lambda_0))(\mathcal{H}).$$

*Proof.* From  $CA - AC = \lambda_0 C$  we get  $AC = C(A - \lambda_0)$ . Put  $B = A - \lambda_0$ . Now take  $\mu \in \rho(B)$ . Then

$$\begin{aligned} (A - \mu)C(B - \mu)^{-1} &= AC(B - \mu)^{-1} - \mu C(B - \mu)^{-1} \\ &= CB(B - \mu)^{-1} - \mu C(B - \mu)^{-1} \\ &= C(B - \mu)(B - \mu)^{-1} = C, \end{aligned}$$

thus  $C(\mathcal{H}) \subseteq (A - \mu)(\mathcal{H})$ . Since  $\mu \in \rho(B)$  was arbitrary, we derive

$$C(\mathcal{H}) \subseteq \bigcap_{\mu \in \rho(B)} (A - \mu)(\mathcal{H}).$$

Proposition 2.1(b) implies now that

$$C(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(B))(\mathcal{H}) = E(\sigma(A) \cap \sigma(A - \lambda_0))(\mathcal{H}).$$

Now suppose that  $D \in N(\delta_A + \lambda_0)$ , hence  $DA = (A - \lambda_0)D = BD$ . Therefore  $D^*B^* = A^*D^*$ . A similar computation as above shows that for  $\mu \in \rho(B^*)$  we have

$$(A^* - \mu)D^*(B^* - \mu)^{-1} = D^*,$$

thus

$$D^*(\mathcal{H}) \subseteq \bigcap_{\mu \in \rho(B^*)} (A^* - \mu)(\mathcal{H}).$$

Since  $\rho(B^*) = \{\bar{\lambda} : \lambda \in \rho(B)\}$ , we get from Proposition 2.1 that

$$\begin{aligned} D^*(\mathcal{H}) &\subseteq \bigcap_{\lambda \in \rho(B)} (A - \lambda)^*(\mathcal{H}) = \bigcap_{\lambda \in \rho(B)} (A - \lambda)(\mathcal{H}) \\ &= E(\sigma(A) \cap \sigma(B))(\mathcal{H}) = E(\sigma(A) \cap \sigma(A - \lambda_0))(\mathcal{H}). \end{aligned}$$

■

The following propositions are of central importance for our investigations.

**Proposition 2.4** *Let  $A$  be a normal operator in  $\mathcal{L}(\mathcal{H})$  and suppose that*

$$\sigma(A) \cap \sigma(A + 2k\pi i) \subseteq \{k\pi i\} \quad \text{for } k = 1, 2, \dots.$$

*If  $k \in \mathbb{N} \setminus \{0\}$ ,  $C \in N(\delta_A + 2k\pi i)$  and  $D \in N(\delta_A - 2k\pi i)$  then*

$$AC = k\pi i C = -CA$$

*and*

$$DA = k\pi i D = -AD.$$

*Proof.* Put  $\lambda_0 = -2k\pi i$ . From  $C \in N(\delta_A - \lambda_0)$  we get from Proposition 2.3 that

$$C(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A + 2k\pi i))(\mathcal{H}).$$

Since  $\sigma(A) \cap \sigma(A + 2k\pi i) \subseteq \{k\pi i\}$ ,

$$E(\sigma(A) \cap \sigma(A + 2k\pi i))(\mathcal{H}) \subseteq E(\{k\pi i\}).$$

From Theorem 12.29 in [8] it follows that  $E(\{k\pi i\}) = N(A - k\pi i)$ . Thus

$$C(\mathcal{H}) \subseteq N(A - k\pi i),$$

hence  $AC = k\pi i C$ . From  $CA - AC = -2k\pi i C$  we conclude that  $CA = -k\pi i C = -AC$ . For  $D \in N(\delta_A - 2k\pi i) = N(\delta_A + \lambda_0)$  we get from Proposition 2.3 that

$$D^*(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A + 2k\pi i))(\mathcal{H}) \subseteq N(A - k\pi i).$$

Thus  $AD^* = k\pi i D^*$ . Therefore  $AD^*x = k\pi i D^*x$  for each  $x \in \mathcal{H}$ . The normality of  $A$  gives  $A^*D^*x = -k\pi i D^*x$ , hence  $A^*D^* = -k\pi i D^*$ , thus  $DA = k\pi i D$ . From  $DA - AD = 2k\pi i$  we derive

$$AD = DA - 2k\pi i = -k\pi i D = -DA. \quad \blacksquare$$

**Proposition 2.5** *Suppose that  $S \in \hat{\Sigma}(\mathcal{H})$  and  $k \in \mathbb{N} \setminus \{0\}$ .*

- (a) *If  $C \in N(\delta_S + 2k\pi i)$  then  $SC = k\pi i C = -CS$ .*
- (b) *If  $D \in N(\delta_S - 2k\pi i)$  then  $DS = k\pi i D = -SD$ .*
- (c) *If  $U \in N(f(\delta_S))$  then  $SU + US = 0$ .*
- (d) *If  $\sigma_p(S) \cap \{n\pi i : n = 1, 2, \dots\} = \emptyset$  then  $N(f(\delta_S)) = \{0\}$ .*

*Proof.* We know that there are operators  $X$  and  $A$  in  $\mathcal{L}(\mathcal{H})$  such that  $X$  is invertible in  $\mathcal{L}(\mathcal{H})$ ,  $A$  is normal and

$$S = X^{-1}AX.$$

Therefore we have  $S - \lambda = X^{-1}(A - \lambda)X$  for each  $\lambda \in \mathbb{C}$  and  $\sigma(S) = \sigma(A)$  and  $\sigma(S - \lambda) = \sigma(A - \lambda)$ . Since  $S \in \hat{\Sigma}(\mathcal{H})$ , we derive that

$$(*) \quad \sigma(A) \cap \sigma(A + 2n\pi i) \subseteq \{n\pi i\}$$

for  $n = 1, 2, \dots$

(a) From  $CS - SC = -2k\pi iC$ , we get

$$CX^{-1}AX - X^{-1}AXC = -2k\pi iC,$$

therefore  $(XCX^{-1})A - A(XCX^{-1}) = -2k\pi i(XCX^{-1})$ . This shows that  $XCX^{-1} \in N(\delta_A + 2k\pi i)$ . From  $(*)$  and Proposition 2.4 we see that

$$AXCX^{-1} = k\pi iXCX^{-1} = -XCX^{-1}A,$$

hence  $SC = k\pi iC = -CS$ .

(b) Similar.

(c) Follows from (a), (b) and Proposition 2.2(d).

(d) Let  $n \in \mathbb{N} \setminus \{0\}$ . Since  $n\pi i \notin \sigma_p(S)$ , we see from (a) that  $N(\delta_S + 2n\pi i) = \{0\}$ . In view of Proposition 2.2(d) it remains to show that  $N(\delta_S - 2n\pi i) = \{0\}$ . Take  $D \in N(\delta_S - 2n\pi i)$  and put  $\tilde{D} = XDX^{-1}$ . As in the proof of (a) we see that  $\tilde{D} \in N(\delta_A - 2n\pi i)$ . From Proposition 2.3 it follows that

$$\tilde{D}^*(\mathcal{H}) \subseteq E(\sigma(A) \cap \sigma(A + 2n\pi i))(\mathcal{H}).$$

By  $(*)$  we get  $\tilde{D}^*(\mathcal{H}) \subseteq E(\{n\pi i\}) = N(A - n\pi i)$ . Since  $\sigma_p(A) = \sigma_p(S)$  and  $n\pi i \notin \sigma_p(S)$ , it follows that  $N(A - n\pi i) = \{0\}$ . Thus  $\tilde{D}^* = 0$ , hence  $D = 0$ . ■

### 3 Proofs

**Proof of Theorem 1.1.** Use Proposition 2.2(g) to see that

$$f(\delta_T)(\delta_T(e^S)) = e^{-T}e^Se^T - e^S = 0,$$

hence  $V = \delta_T(e^S) = e^ST - Te^S \in N(f(\delta_T))$ . Proposition 2.5(c) shows that

$$0 = TV + VT = Te^ST - T^2e^S + e^ST^2 - Te^ST = e^ST^2 - T^2e^S.$$

If  $\sigma_p(T) \cap \{k\pi i : k = 1, 2, \dots\} = \emptyset$ , then by Proposition 2.5(d),  $V = 0$ , thus  $e^ST = Te^S$ . ■

**Proof of Theorem 1.2.** It follows from Theorem 1.1 that  $T^2 e^S = e^S T^2$ . By Proposition 2.2(g) we derive

$$f(\delta_S)(\delta_S(T^2)) = e^{-S} T^2 e^S - T^2 = 0,$$

hence  $U = \delta_S(T^2) = T^2 S - S T^2 \in N(f(\delta_S))$ . Proposition 2.5(c) gives now

$$0 = S U + U S = S T^2 - S^2 T^2 + T^2 S^2 - S T^2 = T^2 S^2 - S^2 T^2.$$

■

**Proof of Theorem 1.3.**

(a) We know from Theorem 1.1 that  $e^S T = T e^S$ , thus

$$f(\delta_S)(\delta_S(T)) = e^{-S} T e^S - T = 0,$$

therefore  $TS - ST \in N(f(\delta_S))$ . Use again Proposition 2.5(c) to see that

$$0 = S(TS - ST) + (TS - ST)S = TS^2 - S^2 T$$

(b) Proposition 2.5(d) gives  $N(f(\delta_S)) = \{0\}$ . Hence  $TS = ST$ .

■

**Proof of Theorem 1.4.** Proposition 2.2(g) shows that

$$\begin{aligned} f(\delta_{T+S})(\delta_{T+S}(e^T)) &= e^{-(T+S)} e^T e^{T+S} - e^T \\ &= e^{-S} e^{-T} e^T e^{T+S} - e^T \\ &= e^{-S} e^S e^T - e^T = 0, \end{aligned}$$

therefore  $U = e^T(T+S) - (T+S)e^T = e^T S - S e^T \in N(f(\delta_{T+S}))$ . Since  $N(f(\delta_{T+S})) = \{0\}$  (Proposition 2.5(d)), it follows that  $U = 0$ , hence  $e^T S = S e^T$ , therefore

$$\begin{aligned} f(\delta_{T+S})(\delta_{T+S}(S)) &= e^{-(T+S)} S e^{T+S} - S \\ &= e^{-S} e^{-T} S e^T e^S - S \\ &= 0. \end{aligned}$$

Hence we see that  $S(T+S) - (T+S)S = ST - TS \in N(f(\delta_{T+S})) = \{0\}$ .

■

**Proof of Theorem 1.5.** Since

$$f(\delta_S)(\delta_S(T)) = e^{-S} T e^S - T = e^{-T} T e^T - T = 0,$$

we have  $TS - ST \in N(f(\delta_S))$ , thus, by Proposition 2.5(c)

$$0 = S(TS - ST) + (TS - ST)S = TS^2 - S^2 T,$$

hence  $TS^2 = S^2 T$ .

If  $\sigma_p(S) \cap \{k\pi i : k = 1, 2, \dots\} = \emptyset$ , we see from Proposition 2.5(d) that  $N(f(\delta_S)) = \{0\}$ , thus  $TS = ST$ .

■

## 4 Corollaries

**Corollary 4.1** *If  $A \in \mathcal{L}(\mathcal{H})$  then*

$$A \text{ is normal} \Leftrightarrow e^A e^{A^*} = e^{A+A^*} = e^{A^*} e^A.$$

*Proof.* The implication „ $\Rightarrow$ “ is clear.

„ $\Leftarrow$ “: Since  $A + A^*$  is selfadjoint,  $\sigma(A + A^*) \subseteq \mathbb{R}$ . Thus  $A + A^* \in \hat{\Sigma}(\mathcal{H})$  and  $\sigma_p(A + A^*) \cap \{k\pi i : k = 1, 2, \dots\} = \emptyset$ . Theorem 1.4 shows now that  $AA^* = A^*A$ . ■

**Corollary 4.2** *If  $A, B \in \mathcal{L}(\mathcal{H})$  are selfadjoint then*

$$A = B \Leftrightarrow e^A = e^B.$$

*Proof.* The implication „ $\Rightarrow$ “ is clear.

„ $\Leftarrow$ “: Since  $A \in \hat{\Sigma}(\mathcal{H})$  and  $\sigma_p(A) \cap \{k\pi i : k = 1, 2, \dots\}$  we see from Theorem 1.5 that  $AB = BA$ . Thus  $A - B$  is selfadjoint and  $e^{A-B} = I$ . Take  $\lambda \in \sigma(A - B)$ . Thus  $\lambda \in \mathbb{R}$  and  $e^\lambda = 1$ , hence  $\lambda = 0$ . This gives  $\sigma(A - B) = \{0\}$ . From  $\|A - B\| = r(A - B) = 0$  we get  $A = B$ . ■

**Corollary 4.3** *Suppose that  $A$  and  $B$  are normal operators in  $\mathcal{L}(\mathcal{H})$  and that  $e^A = e^B$ . Then*

$$A + A^* = B + B^*.$$

*Proof.* Use Corollary 4.1 to see that  $e^{A+A^*} = e^{B+B^*}$ . By Corollary 4.2,  $A + A^* = B + B^*$ . ■

**Corollary 4.4** *If  $A \in \mathcal{L}(\mathcal{H})$  is normal then*

$$A = -A^* \Leftrightarrow e^A \text{ is unitary.}$$

*Proof.* The implication „ $\Rightarrow$ “ is clear.

„ $\Leftarrow$ “: Since  $A$  is normal,

$$e^{A+A^*} = e^A e^{A^*} = e^A (e^A)^* = I = e^0.$$

Now use Corollary 4.2 to derive  $A + A^* = 0$ . ■

For our next result we need the following lemma (see also [8, Theorem 12.37]).



**Lemma 4.1** *If  $T \in \mathcal{L}(\mathcal{H})$  is invertible then there are selfadjoint operators  $A$  and  $B$  in  $\mathcal{L}(\mathcal{H})$  such that*

$$T = e^{iA}e^B, \quad \sigma(A) \subseteq [-\pi, \pi] \quad \text{and} \quad \pi \notin \sigma_p(A).$$

*Proof.* If  $T$  is invertible, so are  $T^*$  and  $T^*T$ . Theorem 12.33 in [8] shows that the positive square root  $(T^*T)^{1/2}$  is also invertible. By [8, Theorem 12.35] there is a unitary  $U \in \mathcal{L}(\mathcal{H})$  with  $T = U(T^*T)^{1/2}$ . Since  $\sigma((T^*T)^{1/2}) \subseteq (0, \infty)$ ,  $\log$  is a continuous real function on  $\sigma((T^*T)^{1/2})$ . Thus the symbolic calculus for selfadjoint operators shows that there is a selfadjoint  $B \in \mathcal{L}(\mathcal{H})$  such that  $(T^*T)^{1/2} = e^B$ . A. Wintner has shown in [16] that there is a selfadjoint  $A \in \mathcal{L}(\mathcal{H})$  such that  $U = e^{iA}$ ,  $\sigma(A) \subseteq [-\pi, \pi]$  and  $\pi \notin \sigma_p(A)$ . ■

**Remarks.**

(1) It is shown in [13] that if  $U \in \mathcal{L}(\mathcal{H})$  is unitary then there is a *unique* selfadjoint operator  $A \in \mathcal{L}(\mathcal{H})$  such that

$$U = e^{iA}, \quad \sigma(A) \subseteq [-\pi, \pi] \quad \text{and} \quad \pi \notin \sigma_p(A).$$

For related results see [9].

(2) Lemma 4.1 shows that an invertible operator in  $\mathcal{L}(\mathcal{H})$  is the product of two exponentials. It is natural to ask whether every invertible operator is an exponential, rather than merely the product of two exponentials. The answer is affirmative if  $\dim \mathcal{H} < \infty$ , as a consequence of [8, Theorem 10.30]. But in general the answer is negative, as one can see from [8, Theorem 12.38]. For normal and invertible operators we have the following results.

**Corollary 4.5** *Suppose that  $T \in \mathcal{L}(\mathcal{H})$  is invertible. The following assertions are equivalent:*

- (a)  *$T$  is normal.*
- (b) *There is some normal  $S \in \mathcal{L}(\mathcal{H})$  such that  $T = e^S$ .*

*Proof.* (b)  $\Rightarrow$  (a): Clear.

(a)  $\Rightarrow$  (b): By Lemma 4.1 there are selfadjoint operators  $A, B \in \mathcal{L}(\mathcal{H})$  such that

$$T = e^{iA}e^B$$

and

$$(1) \quad \sigma(A) \subseteq [-\pi, \pi] \quad \text{and} \quad \pi \notin \sigma_p(A).$$

From  $T^* = e^B e^{-iA}$  and the normality of  $T$  we see that

$$e^{2B} = T^*T = TT^* = e^{iA}e^{2B}e^{-iA},$$

thus

$$(2) \quad e^{2B}e^{iA} = e^{iA}e^{2B}.$$

Use (1) to get

$$(3) \quad iA \in \hat{\Sigma}(\mathcal{H}) \text{ and } \sigma_p(iA) \cap \{k\pi i : k = 1, 2, \dots\} = \emptyset.$$

Since  $2B$  is selfadjoint, we have

$$(4) \quad 2B \in \hat{\Sigma}(\mathcal{H}) \text{ and } \sigma_p(2B) \cap \{k\pi i : k = 1, 2, \dots\} = \emptyset.$$

Therefore it follows from (2), (3), (4) and Theorem 1.3(b) that  $AB = BA$ . Thus  $T = e^{iA+B}$ . Put  $S = iA + B$ . Then  $T = e^S$  and  $S$  is normal.  $\blacksquare$

**Corollary 4.6** *Suppose that  $T \in \mathcal{L}(\mathcal{H})$  is invertible and normal. Then there is a unique normal operator  $S \in \mathcal{L}(\mathcal{H})$  such that*

$$T = e^S, \quad r(S - S^*) \leq 2\pi \quad \text{and} \quad 2\pi i \notin \sigma_p(S - S^*).$$

*Proof.* The proof of Corollary 4.5 shows that there is a normal  $S \in \mathcal{L}(\mathcal{H})$  with  $T = e^S$ ,  $S = iA + B$ , where  $A$  and  $B$  are selfadjoint,  $AB = BA$ ,  $\sigma(A) \subseteq [-\pi, \pi]$  and  $\pi \notin \sigma_p(A)$ . Since  $S - S^* = 2iA$ , we get  $r(S - S^*) \leq 2\pi$  and  $2\pi i \notin \sigma_p(S - S^*)$ . Now suppose that  $R \in \mathcal{L}(\mathcal{H})$  is normal,  $T = e^R$ ,  $r(R - R^*) \leq 2\pi$  and  $2\pi i \notin \sigma_p(R - R^*)$ . Then there are selfadjoint operators  $C, D \in \mathcal{L}(\mathcal{H})$  with

$$R = iC + D \quad \text{and} \quad CD = DC.$$

From  $R - R^* = 2iC$  we see that

$$\sigma(C) \subseteq [-\pi, \pi] \quad \text{and} \quad \pi \notin \sigma_p(C).$$

It follows from  $e^S = e^R$  that  $T^* = e^B e^{-iA} = e^D e^{-iC}$ , thus  $e^{2B} = T^* T = e^{2D}$ . Now use Corollary 4.2 to derive  $B = D$ . From  $e^{iA} e^B = e^{iC} e^D$  we see that

$$e^{iA} = e^{iC}.$$

It is shown in [13] that then  $A = C$  (see Remark (1)). Hence  $S = T$ .  $\blacksquare$

Our final result reads as follows:

**Corollary 4.7** *For  $P \in \mathcal{L}(\mathcal{H})$  the following assertions are equivalent:*

- (a)  $e^{T+P} = e^T$  for all  $T \in \mathcal{L}(\mathcal{H})$ .
- (b) There is some  $k \in \mathbb{Z}$  such that  $P = 2k\pi iI$ .

*Proof.* (b)  $\Rightarrow$  (a): Clear.

(a)  $\Rightarrow$  (b): Take  $T \in \mathcal{L}(\mathcal{H})$  with  $r(T) < \pi$ . Proposition 2.2(e) shows that  $r(\delta_T) < 2\pi$ . Thus, by Proposition 2.2(c),  $M_T = \emptyset$ , hence  $N(f(\delta_T)) = \{0\}$  (Proposition 2.2(a)). From

$$\begin{aligned} f(\delta_T)(\delta_T(T+P)) &= e^{-T}(T+P)e^T - (T+P) \\ &= e^{-(T+P)}(T+P)e^{T+P} - (T+P) \\ &= 0 \end{aligned}$$

we see that  $(T+P)T = T(T+P)$ , hence  $TP = PT$ . Therefore we have shown that

$$(5) \quad TP = PT \text{ for each } T \in \mathcal{L}(\mathcal{H}) \text{ with } r(T) < \pi.$$

Now take  $T \in \mathcal{L}(\mathcal{H})$  with  $r(T) \geq \pi$  and put  $T_0 = \frac{\pi}{2r(T)}T$ . Then  $r(T_0) = \frac{\pi}{2}$ . (5) shows that  $T_0P = PT_0$ . Therefore we have that  $TP = PT$  for all  $T \in \mathcal{L}(\mathcal{H})$ . Thus  $P = \alpha I$  for some  $\alpha \in \mathbb{C}$ . Since  $e^P = I$ ,  $I = e^\alpha I$ , hence  $e^\alpha = 1$ . ■

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